ACM sheaves on a nonsingular quadric hypersurface in $\mathbb{P}^4_k$

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Abstract

We prove that on a nonsingular quadric hypersurface Q in $\mathbb{P}^4_k$ even CI liaison classes of ACM curves are in bijective correspondence with the stable equivalence classes (up to shift in degree) of maximal Cohen-Macaulay graded modules over the coordinate ring R of Q, which in turn, are in bijective correspondence with stable equivalence classes (up to shift in degree) of ACM sheaves on Q. In the situation of a nonsingular quadric hypersurface $Q \in \mathbb{P}^4_k$ work of Knörrer [7] shows that there is a unique nonfree indecomposable MCM module over R. We also describe the ACM sheaf corresponding to this MCM module and give its cohomology table and its Hilbert polynomial.

Keywords: CI liaison, maximal Cohen-Macaulay modules, quadric hypersurface, ACM sheaves.

In classifying algebraic space curves, the special class of Arithmetically Cohen-Macaulay (ACM)curves plays a significant role. C.Peskine and L.Szpiro ([9]) showed that any ACM scheme of codimension two in $\mathbb{P}^4$ is in the CI-liaison class of a complete intersection. In higher codimension this result is no longer true for CI-liaison.

In this work our attention is restricted to the ACM curves that lie on a nonsingular quadric hypersurface Q in $\mathbb{P}^4_k$

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Let X be a complete (connected) arithmetically Gorenstein subscheme of $\mathbb{P}^n_k$ of dimension $r \geq 2$ with a very ample line bundle $\mathcal{L}$. Suppose that $H^1(X, \mathcal{L}^n) = 0$ for all n. (For example X = a smooth hypersurface of $\mathbb{P}^4_k$. Then $h^i(X, O_X(t)) = 0 \;\; \forall t, i = 1, \ldots, r-1$ and X is subcanonical: $w_X \cong O_X(c)$ for some integer c since X is arithmetically Gorenstein.) We start with describing relations between ACM curves on Q and MCM modules on R.

Definition 1. *Let Z be a subscheme of X. Then a resolution*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

where $\mathcal{L} = \oplus O_X(-a_i)$ for some $a_i \in \mathbb{Z}$ and $\mathcal{E}$ is a locally free sheaf on X with $H^1_*(\mathcal{E}) = 0$ is called an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$. For any sheaf $\mathcal{F}$ we denote $H^1(X, \mathcal{F}(t))$ for all $t \in \mathbb{Z}$ by $H^1_*(\mathcal{F})$.

Lemma 1. *Let $Z \subset X$ be locally Cohen-Macaulay equidimensional and $codim_X Z = 2$. Then there exists an $\mathcal{E}$-type resolution $\mathcal{I}_Z$.*

Proof. Let $S(X)$ be the homogeneous coordinate ring of X and $S(Z)$ be the homogeneous coordinate ring of Z, let I_Z be the ideal of Z in $S(X)$, and let

$$(1) \qquad\qquad 0 \longrightarrow E \longrightarrow L \xrightarrow{\alpha} I_Z \longrightarrow 0$$

be a presentation where L is a free graded $S(X)$-module (i.e. $\oplus S(-a_i)$), whose generators are sent by α onto the generators of the homogeneous ideal of Z in $S(X)$. Let $E = \ker \alpha$. By sheafifying one obtains a resolution

$$(2) \qquad\qquad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow \mathcal{I}_Z \longrightarrow 0,$$

where $\mathcal{L}$ is a direct sum of line bundles on X. To show that $\mathcal{E}$ is locally free sheaf on X, consider any closed point $y \in Z$. We know that $\text{proj.dim} \mathcal{O}_{Z,y} = \dim X - \text{depth} \mathcal{O}_{Z,y} = 2$, since $\mathcal{O}_{X,y}$ is regular and Z is locally Cohen-Macaulay. So, $\text{proj.dim} \mathcal{I}_Z = 1$. This implies that $\mathcal{E}$ is locally free. To show that (2) is an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$ we need to prove that $H^1_*(\mathcal{E}) = 0$. From (2) we get the following exact sequence:

$$(3) \qquad 0 \longrightarrow H^0_*(\mathcal{E}) \longrightarrow H^0_*(\mathcal{L}) \longrightarrow H^0_*(\mathcal{I}_Z) \longrightarrow H^1_*(\mathcal{E}) \longrightarrow H^1_*(\mathcal{L}) \longrightarrow \ldots.$$

$H^1_*(\mathcal{L}) = 0$ since $\mathcal{L} = \bigoplus_{i=1}^m \mathcal{O}(-a_i)$. $H^0_*(\mathcal{L}) = L$; $H^0_*(\mathcal{I}_Z) = I_Z$. Therefore sequence (3) is sequence (1). Thus $H^1_*(\mathcal{E}) = 0$ and sequence (3) is an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$. This concludes the proof of the lemma 1 $\qquad\square$

Definition 2. *Two vector bundles $\mathcal{F}$ and $\mathcal{G}$ on X are called* stably equivalent *if*

$$\mathcal{F} \oplus \bigoplus_{i=1}^t O_X(a_i) \cong \mathcal{G} \;\oplus \bigoplus_{i=1}^s O_X(b_i)$$

for some $c, t, d, a_i, b_i \in \mathbb{Z}$.

Theorem 1 ([10] 1.9). *Let X be a nonsingular qadric hypersurface in $\mathbb{P}^4_k$. Then the even liaison classes of curves Z on X are in bijective correspondence with the stable equivalence classes modulo twists of vector bundles $\mathcal{F}$ on X with the property that $H^1_*(\mathcal{F}) = 0$. The correspondence is given by $Z \longrightarrow \mathcal{F}$, where $\mathcal{F}$ is the kernel of an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$.*

We note here that theorem 1 is the statement cited since the nonsingular quadric hypersurface in $\mathbb{P}^4_k$ is a complete connected Gorenstein scheme of dimension at least two. $\mathcal{O}_X(1)$ is a very ample locally free sheaf with $H^i_*(\mathcal{O}_X) = 0$ for $i = 1, 2$. Finally, $\mathcal{L}$-stable equivalence of [10] 1.9 is stable equivalence up to shift in degree defined in this work.

Lemma 2. *Let C be a curve on a nonsingular hypersurface Q in $\mathbb{P}^4_k$. Let*

$$(4) \qquad\qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow \mathcal{I}_C \longrightarrow 0$$

be an $\mathcal{E}$-type resolution of $\mathcal{I}_C$. Then the curve C is ACM if and only if $H^i_*(\mathcal{F}) = 0$ for $i = 1, 2$.

Proof. From theorem 1 we know that $H^1_*(\mathcal{F}) = 0$. Recall that C is ACM is equivalent to $H^1_*(\mathcal{I}_C) = 0$, where $\mathcal{I}_C$ is the ideal sheaf of C. Taking cohomology in (4) we get an exact sequence:

$$H^1_*(\mathcal{F}) \longrightarrow H^1_*(\mathcal{L}) \longrightarrow H^1_*(\mathcal{I}_C) \longrightarrow H^2_*(\mathcal{F}) \longrightarrow H^2_*(\mathcal{L}) \longrightarrow \dots,$$

in which $H^1_*(\mathcal{I}_C) = 0$ since C is ACM curve, and $H^2_*(\mathcal{L}) = 0$ since $\mathcal{L}$ is a direct sum of locally free sheaves. Therefore, if C is ACM, $H^i_*(\mathcal{F}) = 0$ for $i = 1, 2$. Reversing this argument we get the converse of the statement. $\qquad\square$

Definition 3 ([5]). *Let B be a local ring and M be a finitely generated B-module. Then we say M is a* Maximal Cohen-Macaulay *module if $\operatorname{depth} M = \dim B$.*

Definition 4. *On a nonsingular 3-fold Q a locally free sheaf $\mathcal{F}$ with the property that $H^i_*(\mathcal{F}) = 0$ for $i = 1, 2$ is called and* ACM *sheaf.*

Theorem 2. *Let Q be a nonsingular hypersurface in $\mathbb{P}^4_k$. Then the following classes are in bijective correspondence:*

1. *even CI-liaison classes of ACM curves*

2. *stable equivalence classes up to shift in degree of ACM sheaves on Q*

3. *stable equivalence classes up to shift in degree of Maximal Cohen-Macaulay graded modules over the ring $R = k[x_0, \dots, x_4]/I_Q$, where I_Q is the ideal of Q.*

Proof. $(1) \leftrightarrow (2)$ is Theorem 1 plus Lemma 2.

(1) to (3): For an ACM curve C there exists the following resolution:

$$0 \longrightarrow E \longrightarrow L \xrightarrow{\alpha} I_C \longrightarrow 0$$

where I_C is the ideal of C, L is a graded free R-module whose generators are sent by α onto the generators of I_C and $E = \operatorname{Ker} \alpha$. Let $\mathfrak{m}$ be the irrelevant maximal ideal R_+ of R. We associate to the curve C, the module E.

Then, in order to prove that E is MCM we need to show that depth $E = \dim R$. Now, $\dim Q = 3$, therefore $\dim R = 4$; let A_C be the coordinate ring of C. Then $\dim A_C = 2$. Consider the following short exact sequence:

$$0 \longrightarrow I_C \longrightarrow R \longrightarrow A_C \longrightarrow 0$$

From it, taking local cohomology, we will calculate depth I_C:

depth $A_C = 2$ since C is ACM, therefore $H^i_{\mathfrak{m}}(A_C) = 0, i = 0, 1, H^2_{\mathfrak{m}}(A_C) \neq 0$
depth $R = 4$, therefore $H^i_{\mathfrak{m}}(R) = 0, i \leq 3$

Consider

$$0 \longrightarrow H^0_{\mathfrak{m}}(I_C) \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(A_C) \longrightarrow H^1_{\mathfrak{m}}(I_C) \longrightarrow$$
$$\longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(A_C) \longrightarrow H^2_{\mathfrak{m}}(I_C) \longrightarrow H^2_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(A_C) \longrightarrow$$
$$\longrightarrow H^3_{\mathfrak{m}}(I_C) \longrightarrow H^3_{\mathfrak{m}}(R) \longrightarrow H^3_{\mathfrak{m}}(A_C) \longrightarrow \dots .$$

Therefore $H^i_{\mathfrak{m}}(I_C) = 0$ $i = 0, 1, 2$, $H^3_{\mathfrak{m}}(I_C) \neq 0$, therefore depth $I_C = 3$ by the cohomological interpretation of depth.

Similarly consider

$$0 \longrightarrow E \longrightarrow L \longrightarrow I_C \longrightarrow 0$$

Taking local cohomology:

$$0 \longrightarrow H^0_{\mathfrak{m}}(E) \longrightarrow H^0_{\mathfrak{m}}(L) \longrightarrow H^0_{\mathfrak{m}}(I_C) \longrightarrow H^1_{\mathfrak{m}}(E) \longrightarrow$$
$$\longrightarrow H^1_{\mathfrak{m}}(L) \longrightarrow H^1_{\mathfrak{m}}(I_C) \longrightarrow H^2_{\mathfrak{m}}(E) \longrightarrow H^2_{\mathfrak{m}}(L) \longrightarrow H^2_{\mathfrak{m}}(I_C) \longrightarrow$$
$$\longrightarrow H^3_{\mathfrak{m}}(E) \longrightarrow H^3_{\mathfrak{m}}(L) \longrightarrow H^3_{\mathfrak{m}}(I_C) \longrightarrow H^4_{\mathfrak{m}}(E) \longrightarrow H^4_{\mathfrak{m}}(L) \longrightarrow \dots,$$

depth $L = 4$, therefore $H^i_{\mathfrak{m}}(L) = 0, i \leq 3, H^4_{\mathfrak{m}}(E) \neq 0$
depth $I_C = 3$, therefore $H^i_{\mathfrak{m}}(I_C) = 0, i \leq 2, H^3_{\mathfrak{m}}(I_C) \neq 0$

Therefore, $H^i_{\mathfrak{m}}(E) = 0, i = 0, 1, 2, 3, H^4_{\mathfrak{m}}(E) \neq 0$, therefore depth $E = 4$, therefore E is MCM as was to be shown.

(3) to (2): Let E be a graded MCM R-module. Let $\mathcal{F} = \widetilde{E}$ be the associated sheaf over Q. To show that $H^i_\star(\mathcal{F}) = 0$ for $i = 1, 2$ we need the following result:

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Theorem 3 ([5], p.693). *Let R be a graded ring. If M is a graded R-module, then there is a natural exact sequence (where $\mathfrak{m}$ is irrelevant maximal ideal R_+)*

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow H^0_*(\mathcal{E}) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0$$

and $H^i_*(\mathcal{E}) \cong H^{i+1}_{\mathfrak{m}}(M)$ for $i > 0$, where $\mathcal{E}$ is $\widetilde{M}$.

Applying this theorem for $\mathcal{F} = \widetilde{E}$ we obtain:

$$H^i_*\mathcal{F} \cong H^{i+1}_{\mathfrak{m}}(E) \quad i > 0, \quad H^0_*(\mathcal{F}) = E,$$

which means that $H^1_*(\mathcal{F}) = H^2_{\mathfrak{m}}(E); H^2_*(\mathcal{F}) = H^3_{\mathfrak{m}}(E)$. Now E is MCM, therefore depth $E =$ dim $R = 4$, therefore $H^i_{\mathfrak{m}}(E) = 0, i < 4$, therefore $H^2_{\mathfrak{m}}(E) = H^3_{\mathfrak{m}}(E) = 0$, therefore $H^i_*(\mathcal{F}) = 0, i = 1, 2$ as was to be shown.

This concludes the proof of the theorem. $\qquad\square$

Corollary 1. *Let Q be a nonsingular hypersurface in $\mathbb{P}^4_k$. Let C be an ACM curve on Q. If $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{I}_C \to 0$ is an $\mathcal{E}$-type resolution of $\mathcal{I}_C$, then there exists a curve C' with an $\mathcal{E}$-type resolution of $\mathcal{I}_{C'}$ of the form*

$$0 \to \mathcal{E} \oplus \bigoplus_{i=1}^{k} \mathcal{O}(a_i) \to \mathcal{L}' \to \mathcal{I}_{C'} \to 0, \text{ where } \mathcal{L}' = \bigoplus_{j=1}^{k_1} \mathcal{O}(b_j),$$

if and only if C' is CI-linked to C.

If $\quad 0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$ is an $\mathcal{N}$-type resolution of $\mathcal{I}_C$, then there exists a curve C' with an $\mathcal{N}$-type resolution of $\mathcal{I}_{C'}$ of the form

$$0 \to \mathcal{L}' \to \mathcal{N} \oplus \bigoplus_{i=1}^{k} \mathcal{O}(a_i) \to \mathcal{I}_{C'} \to 0, \text{ where } \mathcal{L}' = \bigoplus_{j=1}^{k_1} \mathcal{O}(b_j),$$

if and only if C' is CI-linked to C.

Now as equivalence of CI-biliaison classes of ACM curves and stable equivalence classes up to shift in degree of ACM sheaves is established, we describe ACM sheaves on a nonsingular quadric hypersurface Q in $\mathbb{P}^4_k$.

Theorem 4 ([1]). *Let $k = \overline{k}$, char $k \neq 2$. Let Q be a quadratic form on a vector space V over k considered as an element of $S_2(V^*)$, which is regular in the sense that $R = k[V^*]/Q$ has only an isolated singularity. Then the Maximal Cohen-Macaulay modules over R are such that there are always one or two nonfree indecomposable Maximal Cohen-Macaulay R-modules depending on whether dim V is odd or even; that they both have the same rank and are syzygies of one another when there are two; and that writing $m = \frac{\dim V}{2} - 1$, the rank of their direct sum is 2^m.*

Corollary 2. *In the case of a nonsingular quadric hypersurface in $\mathbb{P}_k^4$ there is only one indecomposable Maximal Cohen-Macaulay module over R and its rank is 2.*

Now we describe this unique MCM module over R.

Proposition 1. *Let L be a line in $Q \subseteq \mathbb{P}_k^4$. Then it has an $\mathcal{E}$-type resolution of the form:*

(5) $$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Q^3(-1) \longrightarrow \mathcal{I}_L \longrightarrow 0$$

Proof. Let I_L be an ideal of a line L in R where R is the homogeneous coordinate ring of Q. Then there is an exact sequence:

$$0 \longrightarrow E \longrightarrow R^3(-1) \xrightarrow{\beta} I_L \longrightarrow 0$$

where $E = \ker \beta$. By sheafifying we obtain the following exact sequence:

(6) $$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Q^3(-1) \longrightarrow \mathcal{I}_L \longrightarrow 0$$

Following the proof of 1 we get that (6) is an $\mathcal{E}$-type resolution of $\mathcal{I}_L$. By 2 this implies that $\mathcal{E}$ is an ACM sheaf on Q. Also note that rank$\mathcal{E} = 2$. $\qquad\square$

Definition 5. *We denote by $\mathcal{E}_0$ the ACM rank 2 sheaf appearing in the $\mathcal{E}$-type resolution of $\mathcal{I}_L$.*

Theorem 5. *The maximal Cohen-Macaulay module E corresponding to $\mathcal{E}_0$ is the unique nonfree indecomposable MCM module on R, and $\mathcal{E}_0$ is the unique (up to shift) indecompsable ACM sheaf on Q that is not isomorphic to a direct sum of line bundles.*

Proof. Let us show that $\mathcal{E}_0$ is not isomorphic to a direct sum of two locally free sheaves.

Suppose $\mathcal{E}_0 = \mathcal{L}_1 \oplus L_2$, where $\mathcal{L}_i$ are locally free sheaves. By 2 this would imply that a line on Q is in the biliaison class of a complete intersection, which is not the case since deg L is odd while degree of a complete intersection on Q is even and CI-liaison preserves parity. Therefore $\mathcal{E}_0$ is not a direct sum of locally free sheaves. Therefore $\mathcal{E}_0$ is an indecomposable ACM sheaf corresponding to an indecomposable MCM module.

By the theorem 4 there is only one isomorphism class of nonfree indecomposable graded MCM modules over R. Therefore $H_*^0(\mathcal{E}_0)$ is the unique MCM given by that theorem. $\qquad\square$

Theorem 5 and corollary 2 imply the follwing:

Corollary 3. *Let Q be a nonsingular quadric hypersurface in $\mathbb{P}_k^4$. Let $\mathcal{E}$ be an ACM sheaf on Q. Then*

$$\mathcal{E} = \bigoplus_{i=1}^{k_1} \mathcal{E}_0(a_i) \oplus \bigoplus_{j=1}^{k_2} \mathcal{O}_X(b_j)$$

for some $a_i, b_j \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}$.

We now calculate dimensions of cohomology groups of $\mathcal{E}_0$ and its Hilbert polynomial.

Proposition 2.

1. *The dimensions of cohomology groups of $\mathcal{E}_0$ are:*

n	0	1	2	3	4	5	6	7	8
$h^0\big(Q,\mathcal{E}_0(n)\big)$	0	0	4	16	40	80	140	224	336

$$h^1_*(\mathcal{E}_0) = h^2_*(\mathcal{E}_0) = 0 \quad and \quad h^0(\mathcal{E}_0(n)) = 0 \ for \ n < 0;$$

2. *$\mathcal{E}_0(n)$ is generated by global section for all $n \geq 2$;*

3. *$\mathcal{E}_0^\vee = \mathcal{E}_0(3)$.*

Proof. 1)   Since $\mathcal{E}_0$ is an ACM sheaf, we have $h^1_*(\mathcal{E}_0) = h^2_*(\mathcal{E}_0) = 0$. Taking cohomology in the short exact sequence

$$0 \longrightarrow \mathcal{I}_Q(n) \longrightarrow \mathcal{O}_{\mathbb{P}^4_k}(n) \longrightarrow \mathcal{O}_Q(n) \longrightarrow 0$$

we arrive at:

$$0 \longrightarrow H^0\big(\mathcal{I}_Q(n)\big) \longrightarrow H^0\big(\mathcal{O}_{\mathbb{P}^4_k}(n)\big) \longrightarrow H^0\big(\mathcal{O}_Q(n)\big) \longrightarrow H^1\big(\mathcal{I}_Q(n)\big) = 0,$$

where $H^1\big(\mathcal{I}_Q(n)\big) = 0$ since Q is ACM. Therefore dimensions are as follows:

n	$h^0\big(\mathcal{I}_Q(n)\big)$	$h^0\big(\mathcal{O}_{\mathbb{P}^4}(n)\big)$		$h^0\big(\mathcal{O}_Q(n)\big)$
< 0	0	0		0
0	0	1		1
1	0	5		5
2	1	15	$\implies$	14
3	5	35		30
4	15	70		55
5	35	126		91
6	70	210		140
7	126	330		204
8	210	495		285

Similarly, from the short exact sequence

$$0 \longrightarrow \mathcal{I}_L \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_L \longrightarrow 0$$

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twisting and taking cohomology we get

$$0 \longrightarrow H^0\big(\mathcal{I}_L(n)\big) \longrightarrow H^0\big(\mathcal{O}_Q(n)\big) \longrightarrow H^0\big(\mathcal{O}_L(n)\big) \longrightarrow H^1\big(\mathcal{I}_L(n)\big) = 0,$$

where $H^1\big(\mathcal{I}_L(n)\big) = 0$ since a line L is an ACM curve. Then dimensions of cohomology groups are:

n	$h^0\big(\mathcal{I}_L(n)\big)$		$h^0\big(\mathcal{O}_Q(n)\big)$	$h^0\big(\mathcal{O}_L(n)\big)$
< 0	0		0	0
0	0		1	1
1	3		5	2
2	11	$\Longleftarrow$	14	3
3	26		30	4
4	50		55	5
5	85		91	6
6	133		140	7
7	196		204	8
8	296		285	9

And similarly from the short exact sequence

(7) $$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_Q^3(-1) \longrightarrow \mathcal{I}_L \longrightarrow 0$$

defining $\mathcal{E}_0$, twisting and taking chomology we get

$$0 \longrightarrow H^0\big(\mathcal{E}_0(n)\big) \longrightarrow H^0\big(\mathcal{O}_Q^3(n-1)\big) \longrightarrow H^0\big(\mathcal{I}_L(n)\big) \longrightarrow 0$$

since $H^1\big(\mathcal{E}_0(n)\big) = 0$ for the Maximal Cohen-Macaulay module $\mathcal{E}_0$. Then dimensions of cohomology groups are:

n	$h^0\big(\mathcal{E}_0(n)\big)$		$h^0\big(\mathcal{O}_Q^3(n-1)\big)$	$h^0\big(\mathcal{I}_L(n)\big)$
< 0	0		0	0
0	0		0	0
1	0		3	3
2	4	$\Longleftarrow$	15	11
3	16		42	26
4	40		90	50
5	80		165	85
6	140		273	133
7	224		420	196
8	336		612	276

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2) Note that $\mathcal{E}_0$ is 2-regular [8, 1.1.4] and thus, by Castelnuovo-Mumford regularity [8, 1.1.5 (3)] $\mathcal{E}_0(n)$ is generated by its global section for all $n \geq 2$.

3) Since $\mathcal{E}_0$ is a locally free sheaf of rank 2, $\mathcal{E}_0^\vee = \mathcal{E}_0(-c_1)$, where c_1 is the first Chern class of $\mathcal{E}_0$. We calculate c_1 from the exact sequence (7):

$$c_1(\mathcal{E}_0) + c_1(\mathcal{I}_L) = c_1\left(\mathcal{O}_Q^3(-1)\right) = -3.$$

L is of codimension 2 on Q, thus $c_1(\mathcal{I}_L) = 0$, wherefrom $c_1(\mathcal{E}_0) = -3$. $\qquad \square$

Proposition 3. *The Hilbert polynomial of $\mathcal{E}_0$ is*

$$P_{\mathcal{E}_0}(n) = 4\binom{n+1}{3}$$

Proof. Consider

$$0 \longrightarrow \mathcal{I}_Q(n) \longrightarrow \mathcal{O}_{\mathbb{P}_k^4}(n) \longrightarrow \mathcal{O}_Q(n) \longrightarrow 0.$$

The Hilbert polynomial of $\mathcal{O}_Q(n)$ is

$$P_Q(n) = \binom{n+4}{4} - \binom{n+2}{4}.$$

From the short exact sequence

$$0 \longrightarrow \mathcal{I}_L \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_L \longrightarrow 0$$

we get that the Hilbert polynomial of $\mathcal{I}_L$ is

$$P_L(n) = \binom{n+4}{4} - \binom{n+2}{4} - n - 1.$$

Thus from the sequence

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_Q^3(-1) \longrightarrow \mathcal{I}_L \longrightarrow 0.$$

we get the Hilbert polynomial of $\mathcal{E}_0$:

$$P_{\mathcal{E}_0}(n) = 3\left[\binom{n+3}{4} - \binom{n+1}{4}\right] - \binom{n+4}{4} + \binom{n+2}{4} + n + 1 = 4\binom{n+1}{3}$$

$\qquad \square$

Lemma 3. *There is an $\mathcal{E}$-type resolution of $\mathcal{E}_0$ of the form*

$$0 \longrightarrow \mathcal{E}_0(-1) \longrightarrow \mathcal{O}^4(-2) \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

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Proof. The dimension of $H^0(\mathcal{E}_0(2))$ is 4 and $\mathcal{E}_0(2)$ is generated by global sections by proposition 2. Thus, there is a surjective map

$$\mathcal{O}^4 \longrightarrow \mathcal{E}_0(2) \longrightarrow 0.$$

Its kernel $\mathcal{E}$ is a rank two ACM sheaf. Thus, by 3 $\mathcal{E}$ is equal to either $\mathcal{E}_0(a)$ or $\mathcal{O}(a) \oplus \mathcal{O}(b)$ for some $a, b \in \mathbb{Z}$. By 2 we must have $\mathcal{E} = \mathcal{E}_0(1)$. Thus, the sequence

$$0 \longrightarrow \mathcal{E}_0(1) \longrightarrow \mathcal{O}^4 \longrightarrow \mathcal{E}_0(2) \longrightarrow 0$$

is exact and it gives an $\mathcal{E}$-type resolution of $\mathcal{E}_0(2)$. $\qquad\square$

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